On Definability of Universal Graphic Automata by Their Input Symbol Semigroups

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Universal graphic automaton $\mathrm{Atm}(G, G')$ is the universally attracting object in the category of automata, for which the set of states is equipped with the structure of a graph G and the set of output symbols is equipped with the structure of a graph G' preserved by transition and output functions of the automata. The input symbol semigroup of the automaton is $S(G, G') = \mathrm{End}\, G \times \mathrm{Hom}(G, G')$. It can be considered as a derivative algebraic system of the mathematical object $\mathrm{Atm}(G, G')$ which contains useful information about the initial automaton. It is common knowledge that properties of the semigroup are interconnected with properties of the algebraic structure of the automaton. Hence, we can study universal graphic automata by researching their input symbol semigroups. For these semigroups it is interesting to study the problem of definability of universal graphic automata by their input symbol semigroups — under which conditions are the input symbol semigroups of universal graphic automata isomorphic. This is the subject we investigate in the present paper. The main result of our study states that the input symbol semigroups of universal graphic automata over reflexive graphs determine the initial automata up to isomorphism and duality of graphs if the state graphs of the automata contain an edge that does not belong to any cycle.

Keywords: generalized Galois theory, automaton, graph, semigroup, isomorphism.

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INTRODUCTION

One of the main trends of the modern algebra is the generalized Galois theory, the basis of which was laid in the research of E. Galois and which is devoted to the study of mathematical objects by means of some derivative algebraic systems connected with the objects in a special way. Various algebraic systems, topological spaces, formal languages and many others were considered as initial mathematical objects, and automorphism groups and endomorphism semigroups of algebraic systems, homeomorphism groups and semigroups of continuous transformations of topological spaces, syntactic monoids of formal languages and many others were considered as derivative algebraic systems. Such studies for automorphism groups of algebraic systems, endomorphism semigroups of graphs, endomorphism rings of modules, and other derivative algebraic systems were very successfully carried out by B. I. Plotkin [1], A. G. Pinus [2,3], L. M. Gluskin [4,5], Yu. M. Vazhenin [6,7], A. V. Mikhalev [8] and many other algebraists. S. Ulam in his well-known book [9] has mentioned the problem of characterization of mathematical objects by their endomorphisms and automorphisms.

The generalized Galois theory also includes studies of structured automata, in which the systems of states and output symbols are objects of some category $\mathbf{K}$ (see, for example, review in [10]). In this case an automaton is considered as an initial mathematical object and the semigroup of input symbols is considered as a derivative algebraic system. For example, L. M. Gluskin, Yu. M. Vazhenin and other authors (see, review in [11]) investigated the endomorphism semigroups of graphs that can be considered as graphic semi-automata (i.e. automata without output symbols, for which the state systems are objects of the graph category $\mathbf{Gr}$). In particular, Yu. M. Vazhenin in his studies [7] considered graphic semi-automata for graphs that contain an edge which does not belong to any cycle. The main result of the paper states that such graphic semi-automata are completely determined (up to isomorphism and duality of graphs) by their semigroups of input symbols.

In this paper we generalize this result for graphic automata, in which the systems of states and output symbols are objects of the graph category $\mathbf{Gr}$. For any graphs G, G', in the category of all graphic automata with the state graph G and the output symbol graph G' there exists the universally attracting object $\mathrm{Atm}(G, G')$ which is called a universal graphic automata over the graphs G, G'. The main result of the paper — Theorem 1 — states that for a reflexive graph G that contains an edge which does not belong to any cycle and a reflexive graph G' the automaton $\mathrm{Atm}(G, G')$ is completely determined (up to isomorphism and duality of graphs) by its input symbol semigroup.

The main result of this paper was announced in [12].

1. PRELIMINARIES

As a preliminary, we introduce some basic terminology from the theory of semigroups, the theory of automata and the theory of graphs that will be used and state the necessary notation. The reader interested in further details may consult, for instance, A. H. Clifford and G. B. Preston [13], B. I. Plotkin [10], A. M. Bogomolov and V. N. Salii [14], F. Harary [15].

Let X, Y be arbitrary sets and let $\rho \subset X \times Y$ be a binary relation. We put $\mathrm{dom}\,\rho = \{x \in X : (\exists y \in Y)(x, y) \in \rho\}$, $\rho^{-1} = \{(y, x) \in Y \times X : (x, y) \in \rho\}$. By *a mapping* of a set X to a set Y, denoted $\varphi : X \to Y$ we mean a single-valued binary relation $\varphi \subset X \times Y$ such that $\mathrm{dom}\,\varphi = X$. For an element $x \in X$, the image of x under φ is denoted by $\varphi(x)$. For a subset $A \subset X$, let us denote $\varphi(A) = \{\varphi(x) \mid x \in A\}$.

A mapping $\varphi : X \to \{x\}$ denoted c_x is called *a constant mapping* of a set X to an element x. A mapping of a set X to itself is called *a transformation* of X. *The identity transformation* of a set X is denoted by Δ_X. A one-to-one mapping $\varphi : X \to Y$ satisfying $\varphi(X) = Y$ is called *a bijection* of X onto Y. In this case *the inverse mapping* $\varphi^{-1} : Y \to X$ is defined by the formula $\varphi^{-1}(y) = x \Leftrightarrow \varphi(x) = y$.

For mappings $\varphi : X \to Y$, $\psi : Y \to Z$, the composition $\varphi\psi : X \to Z$ is defined by the formula $(\varphi\psi)(x) = \psi(\varphi(x))$ for $x \in X$. Denote the Cartesian product $\varphi \times \varphi = \varphi^2$, where $\varphi^2(x_1, x_2) = (\varphi(x_1), (\varphi(x_2))$ for $x_1, x_2 \in X$.

Following [14], by *a graph* we mean an algebraic structure $G = (X, \rho)$, where X is a non-empty set and ρ is a binary relation $\rho \subset X \times X$ that is called *an adjunct relation*. The elements of X and ρ are called *vertices* and *edges*, respectively. The edge $(x, x) \in \rho$ is called *a loop*. Vertices $x, y \in X$ are called *adjacent* if $(x, y) \in \rho$ or $(y, x) \in \rho$. If $e = (x, y)$ is an edge then e is said to join the vertex x with the vertex y. Edges $e_1 = (x_1, y_1), e_2 = (x_2, y_2)$ are called *adjacent* if $y_1 = x_2$. A sequence of adjacent edges

$(x_0, x_1),\ (x_1, x_2),\ \ldots,\ (x_{n-2}, x_{n-1}),\ (x_{n-1}, x_n)$ is called *a path* from the vertex x_0 to the vertex x_n. In case of equality $x_0 = x_n$, the path is called *a cycle*.

A graph $G = (X, \rho)$ is called *reflexive* if its adjunct relation ρ is a reflexive binary relation, i.e. $(x, x) \in \rho$ for any $x \in X$. Further, only reflexive graphs are considered.

For a graph $G = (X, \rho)$ the graph $\widetilde{G} = (X, \rho^{-1})$ is called *the dual graph* of G.

Let $G = (X, \rho)$, $G' = (X', \rho')$ be graphs. A mapping $\varphi : X \to X'$ is called *a homomorphism* of G to G', denoted $\varphi : G \to G'$, if the following condition holds:

$$(\forall x, y \in X)((x, y) \in \rho \Rightarrow (\varphi(x), \varphi(y)) \in \rho').$$

We denote by $\mathrm{Hom}(G, G')$ the set of all homomorphisms of G to G'. Obviously, for any vertex $x \in X'$, satisfying $(x, x) \in \rho'$, the constant mapping $c_x : X \to \{x\}$ is a homomorphism of G to G'.

A homomorphism $\varphi : G \to G'$ is called *an isomorphism* of G onto G' if φ is a bijection of X onto X' and the following condition holds:

$$(\forall x, y \in X)((x, y) \in \rho \Leftrightarrow (\varphi(x), \varphi(y)) \in \rho').$$

An isomorphism of G onto $\widetilde{G'}$ is called *an anti-isomorphism* of G onto G'. Graphs G, G' are called *isomorphic* (respectively, *anti-isomorphic*) if there is an isomorphism (respectively, an anti-isomorphism) of G onto G'.

A homomorphism of a graph $G = (X, \rho)$ to itself is called *an endomorphism* of G. We denote by $\mathrm{End}\, G$ the semigroup of all endomorphisms of G under the composition. Obviously, the identity transformation Δ_X of the vertex set X is an endomorphism of G. Moreover, for any $x \in X$, the constant mapping $c_x : X \to \{x\}$ is an endomorphism of G.

For graphs G, G', $S(G, G')$ let us denote the semigroup $\mathrm{End}\, G \times \mathrm{Hom}(G, G')$ equipped with the following binary operation [10]

$$(\varphi, \psi) \cdot (\varphi_1, \psi_1) = (\varphi\varphi_1, \varphi\psi_1),$$

where $(\varphi, \psi), (\varphi_1, \psi_1) \in \mathrm{End}\, G \times \mathrm{Hom}(G, G')$.

We denote by $Z(G, G')$ the set of all right zeros of the semigroup $S(G, G')$ and by $U(G, G')$ the set of all left identities of the semigroup $S(G, G')$. These sets are respectively defined in the semigroup $S(G, G')$ by the formulas of the semigroup theory $\Phi(x) = (\forall y)(yx = x)$ and $\Psi(x) = (\forall y)(xy = y)$. By analogy with the lemmas 2.1–2.3 [16], it is easy to prove the following results.

Lemma 1. *For any reflexive graphs $G = (X, \rho)$, $G' = (X', \rho')$, the semigroup $S(G, G')$ satisfies the following conditions:*

1) *an element $s \in S(G, G')$ is a right zero of the semigroup $S(G, G')$ if and only if there exist $a \in X$, $b \in X'$ such that $s = (c_a, c_b)$ for the constant mappings $c_a : X \to \{a\}$, $c_b : X \to \{b\}$;*
2) *an element $s \in S(G, G')$ is a left identity of the semigroup $S(G, G')$ if and only if $s = (\Delta_X, \psi)$ for some $\psi \in \mathrm{Hom}(G, G')$.*

Lemma 2. *Let $G = (X, \rho)$, $G' = (X', \rho')$ be reflexive graphs. Then the formula of the semigroup theory*

$$\Sigma(x, y) = \Phi(x) \wedge \Phi(y) \wedge (\forall e)(\Psi(e) \Rightarrow xe = ye)$$

defines the binary relation $\varepsilon_{(G,G')}$ on the semigroup $S(G, G')$, such that the following statements hold:

1) *$\varepsilon_{(G,G')}$ is an equivalence on the set $Z(G,G')$ such that for any elements $s, s_1 \in Z(G,G')$, $s \equiv s_1(\varepsilon_{(G,G')})$ if and only if $s = (c_a, c_x)$, $s_1 = (c_a, c_y)$, for some $a \in X$ and $x, y \in X'$;*
2) *for any right zero $s = (c_a, c_b)$ of the semigroup $S(G,G')$ the equivalence class $\varepsilon_{(G,G')}(s)$ is $\{(c_a, c_x) \mid x \in X'\}$.*

Following [10], by *an automaton* we mean a system $A = (X, S, X', \delta, \lambda)$ consisting of a set of *states* X, a semigroup of *input symbols* S, a set of *output symbols* X', *a transition function* $\delta : X \times S \to X$ and *an output function* $\lambda : X \times S \to X'$ such that

$$\delta(x, st) = \delta(\delta(x, s), t) \quad \text{and} \quad \lambda(x, st) = \lambda(\delta(x, s), t)$$

for any $x \in X$ and $s, t \in S$.

For every $s \in S$, we define the mappings $\delta_s : X \to X$, $\lambda_s : X \to X'$ by the formulas

$$\delta_s(x) = \delta(x, s), \quad \lambda_s(x) = \lambda(x, s),$$

where $x \in X$.

An automaton $A = (X, S, X', \delta, \lambda)$ is said to be *graphic* if its sets X and X' are equipped with structures of graphs $G = (X, \rho)$ and $G' = (X', \rho')$ such that for every $s \in S$ the transition function δ_s is an endomorphism of G and the output function λ_s is a homomorphism of G to G' respectively. In this case we denote the automaton by $A = (G, S, G', \delta, \lambda)$.

By *a homomorphism* of a graphic automaton $A = (G, S, G', \delta, \lambda)$ into a graphic automaton $A_1 = (G_1, S_1, G'_1, \delta_1, \lambda_1)$ we mean an ordered triplet $\gamma = (f, \pi, g)$, consisting of homomorphisms $f : G \to G_1$, $\pi : S \to S_1$, $g : G' \to G'_1$, such that, for any $x \in X$, $s, t \in S$, the following conditions hold

$$\pi(s \cdot t) = \pi(s) \cdot \pi(t), \; f(\delta(x, s)) = \delta_1(f(x), \pi(s)), \; g(\lambda(x, s)) = \lambda_1(f(x), \pi(s)).$$

A homomorphism $\gamma = (f, \pi, g)$ of A to A_1 is called *an isomorphism* of the automaton A onto the automaton A_1 if $f : G \to G_1$, $\pi : S \to S_1$, $g : G' \to G'_1$ are isomorphisms.

An important example of a graphic automaton is given by the following algebraic system

$$\mathrm{Atm}(G, G') = (G, S(G, G'), G', \delta^\circ, \lambda^\circ),$$

where $G = (X, \rho)$, $G' = (X', \rho')$ are graphs and for every $x \in X$, $(\varphi, \psi) \in S(G, G')$ the equalities $\delta^\circ(x, (\varphi, \psi)) = \varphi(x)$, $\lambda^\circ(x, (\varphi, \psi)) = \psi(x)$ hold. It is easy to verify that the automaton $\mathrm{Atm}(G, G')$ satisfies the following universal property: for any graphic automaton $A = (G, S, G', \delta, \lambda)$ there exists a homomorphism $\pi : S \to S(G, G')$ such that $\gamma = (\Delta_X, \pi, \Delta_{X'})$ is a homomorphism of A to $\mathrm{Atm}(G, G')$. For this reason the automaton $\mathrm{Atm}(G, G')$ is called [10] *a universal graphic automaton* over the graphs G, G'.

2. MAIN RESULT

The main result of this paper gives us the following solution of the problem of definability of universal graphic automata by their input symbol semigroups.

Theorem 1. *Let G, G', G_1, G'_1 be reflexive graphs, such that the graph G contains an edge which does not belong to any cycle, and $\mathrm{Atm}(G, G')$, $\mathrm{Atm}(G_1, G'_1)$ be the universal graphic automata over the graphs G, G' and G_1, G'_1 respectively. Then the following conditions are equivalent:*

1) *the graphs* G, G' *are isomorphic to the graphs* G_1, G_1' *or to the dual graphs* $\widetilde{G_1}$, $\widetilde{G_1'}$ *respectively;*
2) *the semigroups of input symbols* $S(G, G')$, $S(G_1, G_1')$ *of the automata* $\mathrm{Atm}(G, G')$, $\mathrm{Atm}(G_1, G_1')$ *are isomorphic;*
3) *the automaton* $\mathrm{Atm}(G, G')$ *is isomorphic to the automaton* $\mathrm{Atm}(G_1, G_1')$ *or to the automaton* $\mathrm{Atm}(\widetilde{G_1}, \widetilde{G_1'})$ *over the dual graphs* $\widetilde{G_1}, \widetilde{G_1'}$.

Proof. Obviously, 3) implies 1), 2). On the other hand, any isomorphisms $f : G \to G_1$, $g : G' \to G_1'$ determine a mapping $\pi : S(G, G') \to S(G_1, G_1')$ by the formula

$$\pi(\varphi, \psi) = (f^2(\varphi), (f \times g)(\psi)),$$

where $(\varphi, \psi) \in S(G, G')$. Indeed, by the definition, the image $f^2(\varphi) = f^{-1}\varphi f$ is an endomorphism of G_1 and the image $(f \times g)(\psi) = f^{-1}\psi g$ is a homomorphism of G_1 to G_1'. It is easy to verify that π is an isomorphism of the semigroup $S(G, G')$ onto the semigroup $S(G_1, G_1')$ such that the ordered triplet $\gamma = (f, \pi, g)$ is an isomorphism of the automaton $\mathrm{Atm}(G, G')$ onto the automaton $\mathrm{Atm}(G_1, G_1')$.

Similarly, any isomorphisms f, g of the graphs G, G' onto the dual graphs $\widetilde{G_1}$, $\widetilde{G_1'}$ determine an isomorphism $\pi : S(G, G') \to S(\widetilde{G_1}, \widetilde{G_1'})$ by the formula $\pi(\varphi, \psi) = (f^2(\varphi), (f \times g)(\psi))$ (here $(\varphi, \psi) \in S(G, G')$) such that the ordered triplet $\gamma = (f, \pi, g)$ is an isomorphism of the automaton $\mathrm{Atm}(G, G')$ onto the automaton $\mathrm{Atm}(\widetilde{G_1}, \widetilde{G_1'})$. Hence, 1) implies 3).

Let us now verify that 2) implies 1). Let π be an isomorphism of the semigroup $S(G, G')$ onto the semigroup $S(G_1, G_1')$. It is well-known that any isomorphism holds the true value of formulas of the semigroup theory. In particular, the isomorphism π preserves the true value of the formulas $\Phi(x)$, $\Psi(x)$ and $\Sigma(x, y)$. It follows that π maps the set $Z(G, G')$ of right zeros of the semigroup $S(G, G')$ onto the set $Z(G_1, G_1')$ of right zeros of the semigroup $S(G_1, G_1')$ and the set $U(G, G')$ of left identities of the semigroup $S(G, G')$ onto the set $U(G_1, G_1')$ of left identities of the semigroup $S(G_1, G_1')$. Moreover, by lemma 2 the Cartesian product π^2 maps the equivalence $\varepsilon = \varepsilon_{(G,G')}$ on the semigroup $S(G, G')$ onto the equivalence $\varepsilon_1 = \varepsilon_{(G_1,G_1')}$ on the semigroup $S(G_1, G_1')$. In view of lemma 1, for any $a \in X$, $a' \in X'$, there exist $a_1 \in X_1$, $a_1' \in X_1'$ such that $\pi(c_a, c_{a'}) = (c_{a_1}, c_{a_1'})$. Hence, π maps the equivalence class $\varepsilon(c_a, c_{a'})$ onto the equivalence class $\varepsilon_1(c_{a_1}, c_{a_1'})$. It follows that the formulas $f(a) = a_1$ and $g_a(a') = a_1'$ determine the mappings $f : X \to X_1$ and $g_a : X' \to X_1'$ $(a \in X)$ such that the equality holds

$$\pi(c_a, c_{a'}) = (c_{f(a)}, c_{g_a(a')}). \tag{1}$$

It is easy to verify that f is a bijection of X onto X_1 and for every $a \in X$, g_a is a bijection of X' onto X_1'.

Consider $(\varphi, \psi) \in S(G, G')$ and for arbitrary $a \in X$ we denote

$$\varphi(a) = b, \ \psi(a) = d. \tag{2}$$

By definition of $S(G, G')$ for any $y \in X'$ the equalities hold

$$(c_a, c_y) \cdot (\varphi, \psi) = (c_a\varphi, c_a\psi) = (c_{\varphi(a)}, c_{\psi(a)}).$$

Then (2) is equivalent to the equality $(c_a, c_y) \cdot (\varphi, \psi) = (c_b, c_d)$.

By definition of the isomorphism π,

$$\pi(c_a, c_y) \cdot \pi(\varphi, \psi) = \pi(c_b, c_d). \tag{3}$$

Let us denote $\pi(\varphi, \psi) = (\varphi', \psi')$. By (1),

$$\pi(c_a, c_y) = (c_{f(a)}, c_{g_a(y)}),\ \ \pi(c_b, c_d) = (c_{f(b)}, c_{g_b(d)}).$$

Then (3) implies

$$(c_{f(a)}, c_{g_a(y)}) \cdot (\varphi', \psi') = (c_{f(b)}, c_{g_b(d)}).$$

On the other hand,

$$(c_{f(a)}, c_{g_a(y)}) \cdot (\varphi', \psi') = (c_{f(a)}\varphi', c_{f(a)}\psi') = (c_{\varphi'(f(a))}, c_{\psi'(f(a))}).$$

Hence,

$$(c_{\varphi'(f(a))}, c_{\psi'(f(a))}) = (c_{f(b)}, c_{g_b(d)}),$$

which is equivalent to the following condition

$$\varphi'(f(a)) = f(b) = f(\varphi(a)),\quad \psi'(f(a)) = g_b(d) = g_{\varphi(a)}(\psi(a)). \tag{4}$$

Then (4) implies that $f\varphi' = \varphi f$, $\varphi' = f^{-1}\varphi f = f^2(\varphi)$ and $\psi' = \psi^{\varphi}$, where the mapping $\psi^{\varphi} : X_1 \to X_1'$ is defined by the formula

$$\psi^{\varphi}(f(a)) = g_{\varphi(a)}(\psi(a))$$

for every $a \in X$.

Thus, for any $(\varphi, \psi) \in S(G, G')$, the following equality holds

$$\pi(\varphi, \psi) = (f^2(\varphi), \psi^{\varphi}). \tag{5}$$

It follows that the mapping $\pi_1 = f^2$ is a bijection of the set $\operatorname{End} G$ onto the set $\operatorname{End} G_1$. Moreover, it is obvious that f^2 preserves the composition. Hence, for any $\varphi, \varphi_1 \in \operatorname{End} G$ the equality $\pi_1(\varphi\varphi_1) = \pi_1(\varphi)\pi_1(\varphi_1)$ holds. Then the mapping π_1 is an isomorphism of the semigroup $\operatorname{End} G$ onto the semigroup $\operatorname{End} G_1$. By the condition of the theorem G is a reflexive graph with an edge $(u_0, v_0) \in \rho$ which does not belong to any cycle. It follows from [7] that the mapping f is an isomorphism of the graph G onto the graph G_1 or onto the dual graphs $\widetilde{G}_1$.

Suppose that f is an isomorphism of G onto G_1. Hence $(f(u_0), f(v_0)) \in \rho_1$ is an edge of the graph G_1, which does not belong to any cycle. Next we show that, for any point $a \in X$ the mapping g_a is an isomorphism of the graph G' onto the graph G_1'. It is easy to verify that for vertices $x_0, y_0 \in X'$ the condition $(x_0, y_0) \in \rho'$ is equivalent to the condition $\psi^2(u_0, v_0) = (x_0, y_0)$ for some homomorphism $\psi \in \operatorname{Hom}(G, G')$. Indeed, if $\psi^2(u_0, v_0) = (x_0, y_0)$ for some homomorphism $\psi : G \to G'$ then $(x_0, y_0) \in \rho'$ by the definition of graph homomorphism. On the other hand, let $(x_0, y_0) \in \rho'$. Define a mapping $\psi : X \to X'$ by the following formula:

$$\psi(u) = \begin{cases} y_0, & \text{if there exists a path from the vertex } v_0 \text{ to the vertex } u, \\ x_0, & \text{otherwise.} \end{cases}$$

We show that ψ is a homomorphism of the graph G to the graph G'. Let $(s,t) \in \rho$. If there exists a path from the vertex v_0 to the vertex s, then by the condition $(s,t) \in \rho$ there exists a path from the vertex v_0 to the vertex t. Then by our definition of ψ the conditions $\psi(s) = \psi(t) = y_0$ hold. Since the graph G is reflexive, the conditions $(\psi(s), \psi(t)) = (y_0, y_0) \in \rho'$ hold. If there is no a path from the vertex v_0 to the vertex s, then by our definition of ψ the condition $\psi(s) = x_0$ holds. Since $\psi(t) \in \{x_0, y_0\}$ and the graph G is reflexive, the condition $(\psi(s), \psi(t)) \in \rho'$ holds.

Thus, in any case the condition $(s,t) \in \rho$ implies $(\psi(s), \psi(t)) \in \rho'$. Hence $\psi \in \mathrm{Hom}(G, G')$.

Since $(f(u_0), f(v_0)) \in \rho_1$ is an edge of the graph G_1, which does not belong to any cycle, it is similar for vertices $x'_0, y'_0 \in X'_1$ that the condition $(x'_0, y'_0) \in \rho'_1$ is equivalent to the condition $\psi^2(f(u_0), f(v_0)) = (x'_0, y'_0)$ for some homomorphism $\psi \in \mathrm{Hom}(G_1, G'_1)$.

Suppose that for vertices $x_0, y_0 \in X'$ the condition $(x_0, y_0) \in \rho'$ holds. It is equivalent to the condition $\psi^2(u_0, v_0) = (x_0, y_0)$ for some $\psi \in \mathrm{Hom}(G, G')$. Hence $(c_a, \psi) \in S(G, G')$ and, as it was proved before, $\pi((c_a, \psi)) = (c_{f(a)}, \psi^{c_a})$ for the homomorphism $\psi^{c_a} \in \mathrm{Hom}(G_1, G'_1)$. By our definition of the mapping ψ^{c_a} for any $x \in X$ the following equalities hold:

$$\psi^{c_a}(f(x)) = g_{c_a(x)}(\psi(x)) = g_a(\psi(x)).$$

In particular, for $u_0, v_0 \in X$ the following equalities hold:

$$\psi^{c_a}(f(u_0)) = g_a(\psi(u_0)) = g_a(x_0), \quad \psi^{c_a}(f(v_0)) = g_a(\psi(v_0)) = g_a(y_0).$$

It follows that the homomorphism $\psi^{c_a} \in \mathrm{Hom}(G_1, G'_1)$ maps the ordered pair $(f(u_0), f(v_0)) \in \rho_1$ to the ordered pair $(g_a(x_0), g_a(y_0))$ and the condition $(g_a(x_0), g_a(y_0)) \in \rho'_1$ holds.

Let us assume the converse case — for some vertices $x'_0, y'_0 \in X'_1$ the condition $(x'_0, y'_0) \in \rho'_1$ holds. It is equivalent to the condition $\psi_1^2(f(u_0), f(v_0)) = (x'_0, y'_0)$ for some homomorphism $\psi_1 \in \mathrm{Hom}(G_1, G'_1)$. Hence, $(c_{f(a)}, \psi_1) \in S(G_1, G'_1)$ and by our definition of the isomorphism $\pi : S(G, G') \to S(G_1, G'_1)$ there exists $(\varphi, \psi) \in S(G, G')$ such that $\pi((\varphi, \psi)) = (c_{f(a)}, \psi_1)$. From the previously proven part follows that $c_{f(a)} = f^2(\varphi)$ and $\psi_1 = \psi^{\varphi}$. Since $c_{f(a)} = f^{-1}\varphi f$ the condition $\varphi = f c_{f(a)} f^{-1} = c_a$ holds. Hence

$$x'_0 = \psi_1(f(u_0)) = \psi^{c_a}(f(u_0)) = g_a(\psi(u_0)), \quad \psi(u_0) = g_a^{-1}(x'_0),$$

$$y'_0 = \psi_1(f(v_0)) = \psi^{c_a}(f(v_0)) = g_a(\psi(v_0)), \quad \psi(v_0) = g_a^{-1}(y'_0)$$

and the homomorphism $\psi \in \mathrm{Hom}(G, G')$ maps the ordered pair $(u_0, v_0) \in \rho$ to the ordered pair $(g_a^{-1}(x'_0), g_a^{-1}(y'_0))$. Thus, $(g_a^{-1}(x'_0), g_a^{-1}(y'_0)) \in \rho'$ and g_a is an isomorphism of the graph G' onto the graph G'_1.

Similarly, if f is an isomorphism of the graph G onto the dual graphs $\widetilde{G_1}$, then all mappings g_a are isomorphisms of the graph G' onto the dual graphs $\widetilde{G'_1}$, since in this case the condition $(u_0, v_0) \in \rho$ is equivalent to the condition $(f(v_0), f(u_0)) \in \rho_1$ and hence $(x_0, y_0) \in \rho'$ is equivalent to $(g_a(y_0), g_a(x_0)) \in \rho'_1$. Therefore, 2) implies 1). $\square$

CONCLUSION

The results obtained shows that universal graphic automata over the graphs from a wide class of graphs are completely determined (up to isomorphism and the duality of graphs) by their input symbol semigroups. Consequently, on the one hand, the characteristics of the input symbol semigroups of universal graphic automata must be defined by the characteristics of the automata, and on the other hand, the characteristics of these semigroups must characterize the automata to same extent. This approach allows us to study the properties of such automata by studying properties of their input symbols semigroups.

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Об определяемости универсальных графических автоматов своими полугруппами входных сигналов

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Универсальный графический автомат $\mathrm{Atm}(G, G')$ — это универсально притягивающий объект в категории автоматов, у которых множество состояний наделено структурой графа G и множество выходных сигналов — структурой графа G', сохраняющимися функциями переходов и выходов автоматов. Полугруппа входных сигналов такого автомата имеет вид $S(G, G') = \mathrm{End}\, G \times \mathrm{Hom}(G, G')$. Она может рассматриваться как производная алгебраическая система математического объекта $\mathrm{Atm}(G, G')$, которая содержит полезную информацию об исходном объекте. Хорошо известно, что свойства такой полугруппы взаимосвязаны со свойствами алгебраической структуры автомата. Следовательно, универсальные графические автоматы можно изучать путем исследования их полугрупп входных сигналов. Для таких полугрупп представляет интерес проблема определяемости универсальных графических автоматов своими полугруппами входных сигналов: при каких условиях полугруппы входных сигналов универсальных графических автоматов будут изоморфны. В данной работе мы исследовали эту проблему. Основной результат нашей работы состоит в том, что универсальные графические автоматы над рефлексивными графами определяются полугруппами своих входных сигналов с точностью до изоморфизма и двойственности графов, если в графе состояний автомата найдется дуга, не входящая ни в один орцикл.

Ключевые слова: обобщенная теория Галуа, автомат, граф, полугруппа, изоморфизм.

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